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# On Landau Damping

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The problem of Landau damping of longitudinal plasma oscillations is investigated by dividing the plasma electrons into two groups. The first group is the main plasma and consists of all electrons with velocities considerably different from the wave velocity while the second group, the resonant electrons, consists of all electrons with velocities near the wave velocity. It is assumed that initially the main plasma has a wave on it while the resonant particles are undisturbed. It is shown that equating the gain in energy of the resonant particles to the loss in energy of the wave gives the correct Landau damping. Only first-order quantities are used in the analysis so that particle trapping which is a nonlinear effect is not involved. The validity of arguments which attribute Landau damping to particle trapping is discussed. The breakdown of the linearized theory and the Galilean invariance of the damping are also investigated.

## INTRODUCTION

THE problem of longitudinal electron oscillations of a plasma has been extensively treated in the literature.[1–5] Many derivations of the phenomenon of Landau damping have been given. Nevertheless one still hears arguments about its reality. It is the purpose of this paper to give a derivation of Landau damping based on a physical model. While nothing basically new is gained, the author feels that some physical insight into the process involved is obtained. In some respects the treatment presented here is close to the discussion given by Bohm and and Gross.[3]

Landau damping has its origin in the strong interaction between a longitudinal plasma wave and those particles whose velocities are nearly equal to its phase velocity. It comes about because these particles in general tend to pick up energy from the wave when they are initially randomly phased with respect to it. In order to compute this effect we will proceed as follows. We divide the electrons into two groups, the main plasma and the resonant particles. The main plasma contains all the electrons except those in a small interval $2\ \Delta V$ about the wave velocity (see Fig. 1). Since the main plasma has no particles traveling at the wave velocity $V_w$, there exist undamped waves of the main plasma which have this velocity. We will assume that such a wave has been set up in the main plasma. Due to interaction of this wave with the resonant particles there will be an exchange of energy between the wave and these particles. If the rate of change of energy of the resonant particles is $dW/dt$, then the rate of loss of energy by the wave to these particles is $-dW/dt$. If we neglect other losses such as those due to harmonic generation, then this will be the total loss of energy by the wave. One can show that the energy associated with harmonic generation or second-order perturbation is fourth order in the amplitude and can thus be neglected. It will be shown shortly that the energy associated with the first-order perturbation is conserved through second order in the wave amplitude.

## BASIC ASSUMPTIONS AND CALCULATIONS

We assume that the plasma is uniform and infinite in extent. The ions are assumed to constitute a fixed uniform neutralizing background. Collisions between individual particles are neglected.

We will investigate only plane longitudinal waves traveling in the $x$ direction. Thus $y$ and $z$ motions can be neglected. Let $N(V)\ dV$ be the number of electrons per unit volume whose $x$ component of

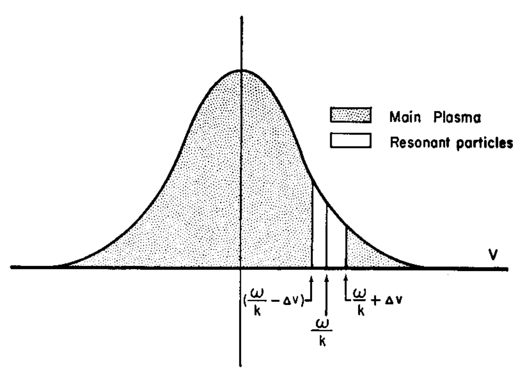


FIG. 1. Division of the plasma into main plasma and resonant particles.

[1] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).
[2] N. G. Van Kamper, Physica **21**, 949 (1955).
[3] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 and 1864 (1949).
[4] R. W. Twiss, Phys. Rev. **88**, 1392 (1952).
[5] J. D. Jackson, J. Nuclear Energy **1**, 171 (1960).

velocity lies between $V$ and $V + dV$ when the plasma is in the undisturbed state. Let $v_1\,(V, x, t)$ and $n_1\,(V, x, t)\,dV$ be the perturbations in velocity and number density for this group of particles. The linearized equations of motion for $v_1$ and $n_1$ are given by (1) and (2):

$$\frac{\partial v_1(V, x, t)}{\partial t} + \frac{V\,\partial v_1(V, x, t)}{\partial x} = \frac{-eE_1}{m} \tag{1}$$

$$\frac{\partial n_1(V, x, t)}{\partial t} + \frac{V\,\partial n_1(V, x, t)}{\partial x} + \frac{N\,\partial v_1(V, x, t)}{\partial x} = 0. \tag{2}$$

Here $E_1$ is the first-order self-consistent electric field produced by all velocity groups and is given by Poisson's equation.

$$\frac{\partial E_1}{\partial x} = -4\pi e \int n_1(V, x, t)\,dV. \tag{3}$$

The velocity $V$ is to be considered constant in (1) and (2).

The first-order quantities $E_1$, $n_1$ and $v_1$ satisfy an energy equation which we shall now derive. It is closely related to the small-amplitude power theorem used in theory of microwave tubes.[6,7] Taking the time derivatives of (3) and making use of (2) we find

$$\frac{\partial^2 E_1}{\partial t\,\partial x} = -4\pi e \int \frac{\partial n_1}{\partial t}\,dV = 4\pi e \int \frac{\partial}{\partial x}(n_1 V + N v_1)\,dV. \tag{4}$$

Integrating over $x$ yields (5) provided we assume there is a place, say $-\infty$, where both $J_1$ and $E_1$ vanish.

$$\partial E_1/\partial t = 4\pi e \int (n_1 V + N v_1)\,dV = -4\pi j_1. \tag{5}$$

From (5) it follows directly that the energy associated with the first-order electric field satisfies the conservation equation (6).

$$\partial(E_1^2/8\pi)/\partial t = -E_1 \cdot j_1. \tag{6}$$

Now consider the kinetic energy of the particles up to second order in the amplitude. It is given by (7):

$$K_1 = \tfrac{1}{2}m \int dV[N v_1^2 + 2N v_1 V + 2 n_1 v_1 V + n_1 V^2 + N V^2]. \tag{7}$$

By taking the derivative of (7) with respect to $t$ and making use of the equations of motion (1) and (2), we find

$$\frac{\partial K_1}{\partial t} = \frac{m}{2}\int dV\left[\frac{-2eE_1}{m}(N v_1 + n_1 V + N V) - 2NV\frac{\partial v_1^2}{\partial x} - 2V^2\frac{\partial(n_1 v_1)}{\partial x} - 3NV^2\frac{\partial v_1}{\partial x} - V^3\frac{\partial n_1}{\partial x}\right]$$
$$= +E_1 \cdot j_1 + E_1 \cdot j_0 - \tfrac{1}{2}m \int dV \frac{\partial}{\partial x}[2NVv_1^2 + 2V^2 n_1 v_1 + 3NV^3 v_1 + V^3 n_1]. \tag{8}$$

The first and second terms on the right-hand side are the rate at which work is being done on the first-order and zero-order currents. The integral on the right-hand side represents a convection of kinetic energy. If $j_0$ is zero the second term vanishes. Also, $E_1$ will in general be made up of terms of the form sin $kx$ and cos $kx$ and will thus average to zero. We will assume this. Further we will assume $n_1$ and $v_1$ are periodic so that integration of the convective terms over a period gives zero. Thus, integrating (8) over one period gives (9):

$$\lambda \frac{\partial}{\partial t}\langle K_1\rangle = \lambda\langle E_1 \cdot j_1\rangle = -\frac{\lambda}{8\pi}\frac{\partial}{\partial t}\langle E_1^2\rangle. \tag{9}$$

Here the brackets $\langle\ \rangle$ indicate average values over a period whose length is $\lambda$. The quantity

$$\lambda(\langle K_1\rangle + (1/8\pi)\langle E_1^2\rangle)$$

is the energy associated with the first-order quantities, $n_1$, $v_1$, and $E_1$ and is conserved.

Let us return now to the problem of Landau damping and assume that the electric field has the form (10)

$$E_1 = E_1 \sin(kx - \omega t). \tag{10}$$

This field is to be produced by a normal mode of the main plasma. On substituting (10) in the equations of motion, (1) and (2), we find that $n_1$ and $v_1$ for the main plasma are given by (11) and (12).

$$v_1(x, t) = -\frac{eE_1}{m(\omega - kV)}\cos(kx - \omega t) \tag{11}$$

$$n_1(x, t) = -\frac{eE_1 N(V) k}{m(\omega - kV)^2}\cos(kx - \omega t). \tag{12}$$

Further, we assume that the electric field is produced by the main plasma and that the resonant particles make a negligible contribution to it. This is equivalent to assuming small damping. On substitution

[6] W. H. Louisell and J. R. Pierce, Proc. Inst. Radio Engrs. **43**, 425 (1955).
[7] P. A. Sturrock, Ann. Phys. (N. Y.) **4**, 306 (1958).

of (10) and (12) in (3), one finds that $\omega$ and $k$ must satisfy the dispersion relation (13).

$$1 = +\frac{4\pi e^2}{m}\int_{mp}\frac{N(V)\,dV}{(\omega - kV)^2}. \qquad (13)$$

Here $mp$ means the integral is to be taken only over the main plasma. Since the resonant particles have been cut out no problem with singular integrals arises here.

We may now compute the wave energy. Equation (14) gives its value per wavelength.

$$W_w = \frac{\lambda E_1^2}{16\pi} + \frac{m}{2}\int_0^\lambda dx\int_{mp} dV \cdot[(N + n_1)(V + v_1)^2 - NV^2]. \qquad (14)$$

Making use of expressions (11) and (12) for $n_1$ and $v_1$ we find for $W_w$

$$W_w = \frac{\lambda}{2}\frac{E_1^2}{8\pi}\left[1 + \omega_p^2\int_{mp}\frac{f(V)(\omega + kV)}{(\omega - kV)^3}\,dV\right]. \qquad (15)$$

Here $f(V)$ is the normalized distribution function

$$f(V) = N(V)\Big/\int_{mp} N(V)\,dV, \qquad (16)$$

and $\omega_p$ is the plasma frequency,

$$\omega_p^2 = 4\pi e^2\int_{mp} N(V)\,dV. \qquad (17)$$

One can convert (15) into (18) by making use of the dispersion relation (13).

$$W_w = \frac{\lambda\omega_p^2E_1^2\omega}{8\pi}\int_{mp}\frac{f(V)\,dV}{(\omega - kV)^3} = \frac{\lambda E_1^2}{8\pi}\Big/\left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right) \qquad (18)$$

$$\lambda = 2\pi/k.$$

We must find the rate of change of energy of the resonant particle. Equations (11) and (12) are solutions for $n_1$ and $v_1$ for the resonant particles as well as for the main plasma. However, they have very large amplitude in the vicinity of the wave velocity. Even if the linearized solutions were valid in this region it would be difficult to set up such a state due to the large amount of order required. We may expect to set up a wave on the main plasma where all particles are perturbed more or less equally, but it would be beyond us to establish the large perturbations of the resonant particles. We will therefore assume that at $t$ equal to zero there is no correlation between the resonant particles and the wave. Thus we will take $n_1(V)$, and $v_1(V)$ equal to zero at $t$ equal zero.

To achieve these initial conditions we must add solutions of the homogeneous equations of motion [(1) and (2) with the right-hand side equal to zero] to solutions (11) and (12). There are two such solutions and they are given by Eqs. (19) and (20).

$$v_1 = v_1(x - Vt),$$
$$n_1 = \left[At + \frac{(1 - A)x}{V}\right]N(V)v_1'(x - Vt) \qquad (19)$$

$$v_1 = 0, \qquad n_1 = n_1(x - Vt). \qquad (20)$$

Here $v_1(x - Vt)$ and $n_1(x - Vt)$ are arbitrary functions of $x - Vt$ and $A$ is an arbitrary constant. The prime indicates differentiation with respect to $(x - Vt)$. Adding solutions (19) and (20) to (11) and (12), so as to satisfy the initial conditions leads to (21) and (22) for the resonant particles.

$$v_1 = -\frac{eE_1}{m(\omega - kV)}\cdot[\cos(kx - \omega t) - \cos k(x - Vt)] \qquad (21)$$

$$n_1 = -\frac{eE_1kN(V)}{m(\omega - kV)^2}\cdot[\cos(kx - \omega t) - \cos k(x - Vt) - (\omega - kV)t\sin k(x - Vt)]. \qquad (22)$$

The last term in expression (22) grows with time. It is due to the fact that a small velocity perturbation at time $t = 0$ will cause a bunching of a stream which increases linearly with time due to the lack of a constraining force.

If (22) is expanded about $V = \omega k$ one finds (23).

$$n_1 \cong \tfrac{1}{2}[eE_1kN(V)t^2]\cos(kx - \omega t). \qquad (23)$$

If $t$ is greater than $\tau$

$$\tau = (m/eE_1k)^{\frac{1}{2}}, \qquad (24)$$

then it is clear from (23) that $n_1$ becomes greater than $N$. Thus for times as great as this solutions (21) and (22) break down.

The time $\tau$ is $1/2\pi$ times the period of oscillation of an electron in the trough of the wave. It depends on the amplitude of the wave and can be made as long as desired by making $E_1$ sufficiently small. However, for wave of experimental interest one should check whether $\tau$ is long or short compared to times of interest.

We may now calculate the energy picked up by the resonant particles. Its value per wavelength is given by (25).

$$W_{\text{res}} = \tfrac{1}{2}m \int_0^\lambda dx \int_{\text{res}} dV \cdot [(N + n_1)(V + v_1)^2 - NV^2]. \qquad (25)$$

After substituting in the values of $n_1$ and $v_1$ given by (21) and (22), we find that

$$W_{\text{res}} = \frac{e^2E_1^2}{m} \lambda \int_{\text{res}} N(V)\, dV \left\{ \frac{\sin^2 [\frac{1}{2}(\omega - kV)]t}{(\omega - kV)^2} + \frac{2kV}{(\omega - kV)^3} [\sin^2 [\tfrac{1}{2}(\omega - kV)]t - \tfrac{1}{4}(\omega - kV)t \sin (\omega - kV)t] \right\}. \qquad (26)$$

The first term of the integrand of (26) comes from the term $Nv_1^2$. The wave generates a motion of the particles relative to the mean velocity or center of mass velocity. The kinetic energy of this motion is what is given by $Nv_1^2$. This term is always positive. The last two terms of the integrand come from the term $2Vv_1n_1$. They give the change in kinetic energy due to a change in the mean velocity of a stream. This term is positive for particles traveling at less than the wave velocity and negative for particles traveling faster than the wave velocity. Particles traveling slower than the wave velocity tend to be accelerated while those traveling faster than the wave velocity tend to be decelerated.

If the second term of the integrand of (26) is integrated by parts, one obtains

$$\begin{aligned} &\frac{e^2E_1\lambda}{m} \int_{\omega/k-\Delta V}^{\omega/k+\Delta V} dV\, \frac{2kN(V)V}{(\omega - kV)^3} \sin^2 [\tfrac{1}{2}(\omega - kV)t] \\ &= 2\, \frac{e^2E_1^2\lambda}{m} \frac{\overline{N(\omega/k)}}{k^2\, \Delta V} \sin^2 \tfrac{1}{2}\, k\, \Delta Vt \\ &\quad - \frac{e^2E_1^2\lambda}{m} \int_{\omega/k-\Delta V}^{\omega/k+\Delta V} \frac{dV}{(\omega - kV)^2} \\ &\quad \cdot \{[N'(V)V + N(V)] \sin^2 [\tfrac{1}{2}(\omega - kV)t] \\ &\quad - \tfrac{1}{2}[N(V)kVt] \sin (\omega - kV)t\}, \end{aligned} \qquad (27)$$

where $\overline{N(\omega/k)}$ is some value that $N(V)$ takes on in the interval

$$(\omega/k - \Delta V) \leq V \leq (\omega/k + \Delta V).$$

It will be approximately $N(\omega/k)$. The same singularity appears in the integrands on the right and left sides of (27). Since the singularity cancels when (27) is substituted in (26), no trouble arises. The first term on the right-hand side of (27) behaves like $1/\Delta V$. We will assume that $1/\Delta V$ is sufficiently large so that this term can be neglected. For this approximation to be valid we require that this term be small compared to the wave energy given by (18). This is true if

$$\left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right) \frac{8\pi e^2 N(\omega/k)}{mk^2\, \Delta V} \sin^2 k\, \Delta Vt \ll 1,$$

or roughly if

$$2\omega_p^2 N(\omega/k)/k^2\, \Delta VN \ll 1, \qquad (28)$$

where $N$ is the total number density for the electrons. For a Maxwell distribution this reduces to

$$1 \equiv \frac{2}{\pi} \frac{\omega_p^2}{k^2\, \Delta V V_T} \exp [-(V^2/2V_T^2)].$$

Here $V_T$ is the thermal velocity of an electron. If $\omega/k$ is large compared to $V_T(\omega/k = 3V_T$ is already large), $\Delta V$ may be chosen much smaller than $\omega/k$ and yet large enough to satisfy (28).

The second and third terms in the integrand on the right-hand side of (27) cancel the first and third terms in the integrand in (26) and thus (26) reduces to (29)

$$W_{\text{res}} = -\frac{e^2E_1\lambda}{m} \int_{\omega/k-\Delta V}^{\omega/k+\Delta V} \frac{N'(V)V}{(\omega - kV)^2} \cdot \sin^2 [\tfrac{1}{2}(\omega - kV)t]\, dV. \qquad (29)$$

If again we assume $\Delta V$ is sufficiently large so that we can treat it as though it were infinite we can integrate (29) and we find

$$W_{\text{res}} = -(\pi e^2 E_1^2 \lambda\omega t/2mk^2)N'(\omega/k). \qquad (30)$$

Taking the time derivative of (30) and equating it to the negative of the time derivative of the wave energy we obtain

$$\frac{dW_w}{dt} = \frac{\lambda\, d(E_1^2)/dt}{8\pi} \left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right)^{-1} = e^2E_1^2 \frac{\pi\omega\lambda}{2mk^2} N'\left(\frac{\omega}{k}\right)$$

or

$$\frac{d(E_1^2)}{dt} = \left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right) \frac{4\pi^2\omega}{mk^2} e^2 N'\left(\frac{\omega}{k}\right)E_1 = \pi\left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right) \frac{\omega_p^2\omega}{k} f'\left(\frac{\omega}{k}\right)E_1^2.$$

From (30) we find $dE_1/dt$ to be given by (30).

$$\frac{dE_1}{dt} = \frac{\pi\omega}{2} \left(1 - \frac{k}{\omega}\frac{d\omega}{dk}\right) \frac{\omega_p^2}{k^2} f'\left(\frac{\omega}{k}\right)E_1. \qquad (31)$$

Equation (31) is the damping one obtains by Landau's formalism.

### GALILEAN INVARIANCE

The treatment of Landau damping that has been given is nonrelativistic. The rate of damping, thus, should be invariant under a Galilean transformation. Both the wave energy $W_w$ and the resonant particle energy depend on the frame of reference. The only quantity which depends on the frame in expressions (18) and (28) for $W_w$ and $W_{res}$ is $\omega$. Thus both these quantities transform like $\omega$. The ratio of the wave energy to the resonant particle energy is independent of the frame and thus the damping given by (30) is also.

The Galilean invariance makes clear the resolution of the following paradox. One often hears it said that particles which are traveling faster than a wave tend to give energy to the wave while particles traveling slower than the wave tend to take up energy from it. That this is so, is clear from the fact that only the term in $N'$ remained when expression (26) for $W_{res}$ was integrated. Thus, if there are more particles moving slower than the wave than there are particles moving faster than the wave, there will be a net absorption of energy and the wave will be damped. However, if this is true one might say, let me go to a frame which is moving relative to the rest frame of the electron with a velocity which is greater than the phase velocity of the wave. Then in this frame there are more particles moving faster than the wave than there are ones moving slower than it. Thus in this frame the wave should gain energy and grow.

The answer to this paradox is the following. The wave energy depends on the frame, and is negative in the moving frame contemplated above. That is, the electrons have less energy than they would have if the wave were not present. Thus even though the wave gains energy from the resonant particles in this frame, it loses negative energy and hence, damps out.

### TRAPPING

There have been a number of explanations of Landau's damping based on the trapping of particles[5] in the wave troughs. These usually obtain the damping to within a numerical factor. Such calculations generally compute the change in a particle's energy upon becoming trapped. Then they estimate the trapping time and take the change in energy divided by the trapping time as the rate at which the trapped particles pick up energy.

Such an explanation relies upon a nonlinear effect to explain a linear phenomenon. No such recourse should be needed. As we have seen, the phenomenon

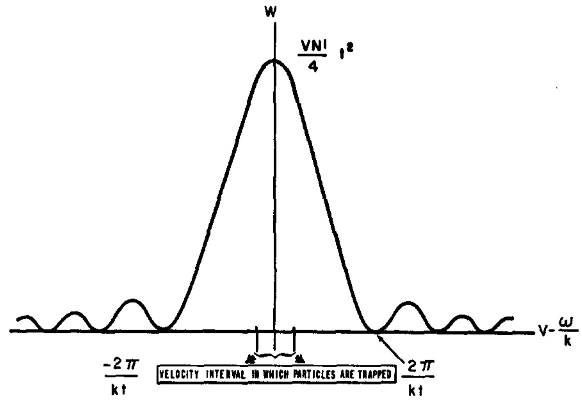


FIG. 2. Contribution of different velocities to the absorbed energy.

is explainable simply in terms of the rate of exchange of energy between the wave and those particles traveling near its phase velocity, and it appears already in the linearized theory. Since trapping does not appear in the linearized theory it cannot be the cause of the effect.

All particles with velocities near the wave velocity, both trapped and untrapped, contribute to the absorption of energy. If one makes a plot of the integrand appearing in Eq. (29) one gets a curve like that shown in Fig. 2. This plot shows which particles or velocities have contributed to the absorbed energy. It does not show the actual energy absorbed per velocity interval $dV$, since that is given by the integrand in Eq. (26), a much more complicated function. However, it does give a good idea of which particles are important. Figure 2 is a typical resonance curve whose height varies as $t^2$ and whose width varies as $1/t$. Most of the absorbed energy is in the central peak. Up to the time that the width of the main peak is of the order of $k\,\delta V$, where $\delta V$ in the velocity interval over which particles are trapped by the wave,

$$\delta V = (2eE_1/mk)^{\frac{1}{2}}, \qquad (32)$$

trapping plays a very small role in the damping process. The trapping time, or time required for an electron to perform one oscillation in the trough of the wave, is given by (33) and is essentially the time $\tau$ [given by Eq. (24)], at which the linearized solution breaks down.

$$t = \sqrt{2}\,\pi(m/eE_1k)^{\frac{1}{2}} = \sqrt{2}\,\pi\tau. \qquad (33)$$

For times greater than $\tau$ trapping will be dominant if the wave has not already died out.

We can now see why the estimate based on the trapping gives the right answer. There is a maximum

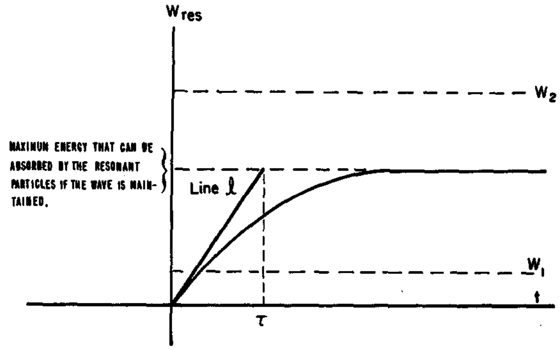


FIG. 3. Energy absorbed vs time.

energy that the resonant particles can absorb if the wave is maintained; since an equilibrium state will ultimately be reached. It is roughly the energy absorbed at the time the linearized solution breaks down. Thus the energy absorbed vs time will have the appearance of Fig. 3 (assuming the wave is maintained). At the time the linear theory breaks down most of the energy that has been absorbed by the resonant particles is contained in those particles which are trapped. Thus the energy of the trapped particles roughly gives the saturation energy which the resonant particles can absorb. By drawing the straight line $l$ (see Fig. 3), which goes through the origin and intersects the saturations energy at $t = \tau$, one obtains a good approximation to the linearized theory for small times and this is what the trapped particles calculations give.

If the wave energy is much smaller than the saturation energy, then the wave will die out long before trapping becomes important. On the other hand, if the wave energy is large compared to the saturation energy trapping will be important. Put another way, if the Landau damping time is short compared to the trapping time $\tau$, the linearized theory gives the correct damping, while if it is long compared to the trapping time the damping will be modified and the Landau theory is incorrect.


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